PHYSICAL REVIEW B **79**, 100504(R) (2009)

Theory of viscoelastic behavior of solid ^{4}He

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(Received 11 February 2009; published 6 March 2009)

Over the last five years several experimental groups have reported anomalies in the temperature dependence of the period and amplitude of a torsional oscillator containing solid ^{4}He. We model these experiments by assuming that ^{4}He is a viscoelastic solid—a solid with frequency-dependent internal friction. We find that while our model can provide a quantitative account of the dissipation observed in the torsional oscillator experiments, it only accounts for about 10% of the observed period shift, leaving open the possibility that the remaining period shift is due to the onset of superfluidity in the sample.

DOI: 10.1103/PhysRevB.79.100504 PACS number(s): 67.80.bd, 46.35.+z

I. INTRODUCTION

In 2004 Kim and Chan[1,2] reported anomalies in the resonant period of a torsional oscillator (TO) containing solid ^{4}He. With exquisite sensitivity they detected a reproducible decrease in the oscillator period upon lowering the temperature below 200 mK. Subsequent experiments in several laboratories[3–11] have now reproduced these results. The size of the effect depends on the sample preparation, pressure, and concentration of ^{3}He impurities. References 12 and 13 review the recent theoretical and experimental works.

A natural interpretation of the TO anomalies is the onset of the elusive and long-anticipated supersolid phase of matter.[14–16] In a supersolid, superfluidity coexists with the crystalline order of a solid; one expects a supersolid to exhibit superflow, much like a superfluid. Leggett[16] proposed that the superflow is best detected by searching for "nonclassical rotational inertia": a superfluid condensate would remain at rest and not participate in rotation, and the resulting mass decoupling would reduce the rotational inertia and would decrease the resonant period of oscillation. While compelling, the supersolid interpretation of these experiments has yet to be corroborated by other measurements, such as the response to pressure differences.[17] Moreover, Day and Beamish[18] reported a pronounced increase in the shear modulus of ^{4}He at temperatures below 200 mK, with a dependence on measurement amplitude and ^{3}He impurity concentration similar to the TO anomalies. Their results suggest that changes in the shear modulus might be intimately related to the TO anomalies.

This work presents a detailed discussion of the mechanical response of a TO containing a viscoelastic solid. We build on earlier work by Nussinov *et al.*,[19] who correctly identified a "back-action" term in the TO equation of motion that represents the dynamical effect of the solid helium on the torsion cell. However, in contrast to Nussinov *et al.*[19] we find no need to assume that the solid helium behaves as a glass. Instead, with a few carefully stated assumptions, we find that we can model the solid helium as a classical viscoelastic solid—i.e., a solid with internal friction. The TO period shift and dissipation peak are naturally related to the real and imaginary parts of the frequency-dependent shear modulus of the solid. We use our results to fit the dissipation peak reported by Clark *et al.*[5] and extract a temperature-dependent time scale $\tau(T)$ from the data. With all of the phenomenological parameters determined, we find that we are only able to account for about 10% of the period shift reported in Ref. 5, leaving open the possibility that the remaining shift is due to the onset of superfluidity (or supersolidity) in these samples.

II. THE MODEL

Following Nussinov *et al.*,[19] we assume that the empty torsion cell is perfectly rigid, with a moment of inertia $I_{\rm osc}$ about its rotation axis. For small angular displacements θ the torsion rod provides a restoring torque $-\alpha\theta$, with α as the torsional spring constant. There is also a damping torque $-\gamma_{\rm osc} d\theta/dt$, with $\gamma_{\rm osc}$ as a temperature-dependent dissipation coefficient. The cell is driven by an external driving torque of $\tau_{\rm ext}(t)$. Finally, the solid helium inside the torsion cell exerts a moment $M(t)$ on the cell. The equation of motion for the cell is then

$$\left(I_{\rm osc}\frac{d^2}{dt^2}+\gamma_{\rm osc}\frac{d}{dt}+\alpha\right)\theta(t)=\tau_{\rm ext}(t)+M(t), \tag{1}$$

with $M(t)=\int dt' g(t-t')\theta(t')$ for a linear system that is invariant under time translations. We can Fourier transform the equation of motion to find the response function $\chi(\omega)=\theta(\omega)/\tau_{\rm ext}(\omega)$ of the TO:

$$\chi^{-1}(\omega)=-I_{\rm osc}\omega^2-i\gamma_{\rm osc}\omega+\alpha-g(\omega). \tag{2}$$

The response function is of fundamental importance in interpreting the TO experiments as its poles give the resonant frequencies and their quality factors.

The complex back-action term $g(\omega)$ contains *all* of the information about the dynamical response of the solid helium, and modeling this quantity is the focus of the remainder of this Rapid Communication. Before delving into calculational details we can make a few general statements about $g(\omega)$. First, causality requires $g(z)$ to be analytic in the upper half of the complex z plane, so the real and imaginary parts of $g(z)$ satisfy Kramers-Kronig relations for $z=\omega$. Second, if the helium behaves as a perfectly rigid, normal solid, $g(\omega)=I_{\rm He}\omega^2$, with $I_{\rm He}$ as the rigid-body moment of inertia of the helium. As we will show below, a finite shear modulus produces corrections to this result which vanish with a *higher*

power of ω at low frequencies. Therefore $g(\omega=0)=0$; the backaction only modifies *dynamical* quantities. At low frequencies the "glassy" model for $g(\omega)$ proposed by Nussinov *et al.*[19] [see Eq. (20) of Ref. 19] goes to a constant and induces an unphysical shift in the static spring constant α.

To calculate the back-action term we first need to model the properties of the helium in the torsion cell. To keep the description as general (and simple) as possible we model the helium as an elastic continuum, with an equation of motion

$$\rho\partial_t^2 u_i = \partial_j \sigma_{ij}, \tag{3}$$

where ρ is the density, u_i is the ith component of the displacement field, and σ_{ij} is the stress tensor. The stress tensor has both a reversible piece σ_{ij}^r and a dissipative piece σ_{ij}^d, with $\sigma_{ij}=\sigma_{ij}^r+\sigma_{ij}^d$.[20] For a linear medium the reversible stress is linearly related to the strain tensor $u_{ik}=(\partial_k u_i+\partial_i u_k)/2$ by $\sigma_{ij}^r=\lambda_{ijkl}u_{kl}$, where λ_{ijkl} is a fourth-rank tensor of elastic coefficients. For a uniaxial crystal such as the hcp phase of ^{4}He there are five independent components of λ_{ijkl}; however, to simplify the analysis we assume that the helium can be modeled as an isotropic elastic solid, with $\sigma_{ij}^r=2\mu_0 u_{ij}+\lambda_0\delta_{ij}u_{kk}$, where λ_0 and μ_0 are the Lamé coefficients (with μ_0 as the shear modulus). The dissipative part of the stress tensor, which describes the *internal friction* of the solid, must be odd under time reversal and can only depend on gradients of the velocity $v_i=\partial_t u_i$.[20] For a linear medium $\sigma_{ij}^d=\eta_{ijkl}v_{kl}$, where η_{ijkl} is the viscosity tensor of the solid and $v_{ik}\equiv(\partial_t v_k+\partial_k v_i)/2$. Again assuming an isotropic medium, $\sigma_{ij}^d=2\eta v_{ij}+(\zeta-2\eta/3)\delta_{ij}v_{kk}$, with η and ζ as the shear and bulk viscosities of the solid. With these simplifications the Fourier transformed equation of motion becomes

$$-\rho\omega^2\mathbf{u} = B(\omega)\,\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u}) - \mu(\omega)\,\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{u}, \tag{4}$$

with $B(\omega)=\lambda_0+2\mu_0-i\omega(\zeta+4\eta/3)$ and $\mu(\omega)=\mu_0-i\omega\eta\equiv\mu_0(1-i\tau\omega)$, with time scale $\tau=\eta/\mu_0$. In passing we note that this model, known as the Kelvin-Voigt model, is among the simplest of viscoelastic models—a single "spring" (the elasticity) is in parallel with a single "dashpot" (the viscosity). More elaborate models, involving series and parallel combinations of springs and dashpots, can produce a shear modulus $\mu(\omega)$ with a more complicated frequency dependence. For an example of a similar analysis for colloidal crystals, see Ref. 21.

The next step is to determine the response of the helium inside the torsion cell to the rotation of the cell. For simplicity we will present results for a torsion cell that is an infinitely long cylinder of radius R; the generalizations to an annular geometry or a cylinder of height h are straightforward.[22] If we assume the helium is in perfect contact with the walls of the torsion cell (no-slip boundary conditions) and the torsion cell oscillates about its azimuthal axis with a frequency ω and amplitude θ_0, then the induced displacements in the helium have the form $\mathbf{u}=u_\theta(r)e^{-i\omega t}\hat{\theta}$. Substituting this into Eq. (4) and solving the differential equation with the no-slip boundary condition $u_\theta(r=R)=R\theta_0$, the solution that is finite at $r=0$ is

$$u_\theta(r) = R\theta_0\frac{J_1(kr)}{J_1(kR)}, \tag{5}$$

where $k^2=\omega^2\rho/\mu(\omega)$ and $J_1(z)$ is the Bessel function of order 1. In this geometry the torsion cell only induces shearing displacements in the helium.

The final step of the calculation is to determine the moment that the oscillating helium exerts back on the torsion cell. The only nonvanishing component of the stress tensor is $\sigma_{\theta r}=\mu(\omega)(\partial_r-1/r)u_\theta(r)$. Evaluating this on the surface of the cylinder, integrating over the area of the cylinder to obtain a force, and then multiplying by the radius to obtain a torque, we find the moment

$$M(t) = -\theta_0\omega^2 I_{\mathrm{He}}\frac{4J_2(kR)}{kRJ_1(kR)}e^{-i\omega t}, \tag{6}$$

where $I_{\mathrm{He}}=\pi\rho hR^4/2$ is the rigid-body moment of inertia for the helium inside the torsion cell. In terms of the back-action term $g(\omega)$ defined in Eq. (2),

$$g(\omega) = I_{\mathrm{He}}\omega^2 + I_{\mathrm{He}}\omega^2\tilde{g}(kR), \tag{7}$$

where the function

$$\tilde{g}(x) = \frac{4J_2(x)}{xJ_1(x)} - 1 \tag{8}$$

is the correction to the rigid-body result due to the finite shear modulus of the helium.

To simplify our result, we note that for a typical TO the speed of transverse sound $c_T=\sqrt{\mu_0/\rho}\sim 270$ m/s, the frequency $f\sim 10^3$ s^{-1}, and the radius $R\sim 0.5$ cm so that $|k|R=2\pi fR/c_T\sim 0.1$. Therefore we can safely expand Eq. (8) for small x, with the result $\tilde{g}(x)\simeq x^2/24$; then Eq. (2) becomes

$$\chi^{-1}(\omega) \simeq -I_{\mathrm{tot}}\omega^2 - i\gamma_{\mathrm{osc}}\omega + \alpha - \frac{\rho R^2\omega^4 I_{\mathrm{He}}F(R/h)}{24\mu(\omega)}, \tag{9}$$

where $I_{\mathrm{tot}}=I_{\mathrm{He}}+I_{\mathrm{osc}}$. We see that the last term in Eq. (9) is the correction due to a finite shear modulus; for a perfectly rigid body, $\mu\to\infty$ and this term vanishes. Also, in this last term we have introduced a function $F(x)$ to describe the effect of a finite cylinder height h (Ref. 22); this function only depends on the aspect ratio $x=R/h$, with the explicit form

$$F(x) = -\frac{192}{\pi^4 x^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^4}\tilde{g}[i(2m-1)\pi x]. \tag{10}$$

For the infinite cylinder $F(0)=1$, and more generally $0\le F(x)\le 1$. For large x, $F(x)\simeq 2/x^2-744\zeta(5)/\pi^5x^3$, with $\zeta(n)$ being the Riemann-Zeta function. In the particular case $h=R$, $F(1)=0.527$.

We now examine the effect of the viscoelasticity of the helium on the period and Q factor of the oscillator by finding the poles of the response function [Eq. (9)] (our analysis is similar to the procedure performed by Nussinov *et al.*[19]). Since $|k|R=\omega_0 R/c_T\sim 0.1$ and $I_{\mathrm{He}}/I_{\mathrm{tot}}\sim 10^{-3}$, we can treat the fourth term in Eq. (9) as perturbation about the rigid-body behavior, which has a resonant frequency $\omega_0=\sqrt{\alpha/I_{\mathrm{tot}}}$ and dissipation $Q_0^{-1}=\gamma_{\mathrm{osc}}/\sqrt{I_{\mathrm{tot}}\alpha}$. Expanding the poles about ω_0 and Q_0^{-1} and recalling that $\mu(\omega)=\mu_0(1-i\tau\omega)$, we obtain the

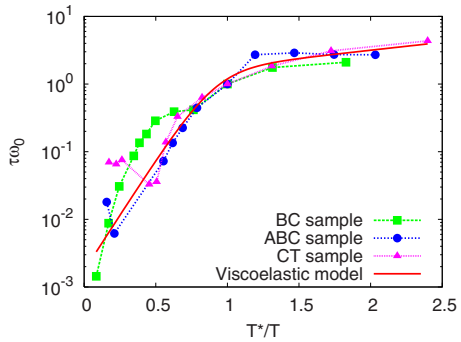

FIG. 1. (Color online) Relaxation times derived from $\Delta Q^{-1}(T)$ of BeCu TO of Clark *et al.* (Ref. 5). Squares (green) represent the blocked capillary (BC) sample, circles (blue) represent the annealed blocked capillary (ABC) sample, and triangles (pink) represent the constant temperature (CT) sample. The heavy line (red) was found by fitting to all three samples and using $\tau(T)\omega_0 = a\exp(bT^*/T)/[1+c\exp(dT^*/T)]$ with fitting parameters $a=1.75\times10^{-3}$, $b=7.55$, $c=1.62\times10^{-3}$, and $d=7.01$.

fractional period shift $\Delta P/P_0=(P-P_0)/P_0$ and the shift in the dissipation $\Delta Q^{-1}=Q^{-1}-Q_0^{-1}$:

$$\frac{\Delta P}{P_0} = A\ \mathrm{Re}\left[\frac{\mu_0}{\mu(\omega_0)}\right] = A\frac{1}{1+(\tau\omega_0)^2}, \tag{11}$$

$$\Delta Q^{-1} = 2A\ \mathrm{Im}\left[\frac{\mu_0}{\mu(\omega_0)}\right] = 2A\frac{\tau\omega_0}{1+(\tau\omega_0)^2}, \tag{12}$$

where the dimensionless amplitude A is given by (recall $c_T=\sqrt{\mu_0/\rho}$)

$$A = \frac{F(R/h)}{48}\frac{I_{\mathrm{He}}}{I_{\mathrm{tot}}}\left(\frac{\omega_0 R}{c_T}\right)^2. \tag{13}$$

While A depends on the material parameters and the sample geometry, *it is independent of the relaxation time* τ. For a typical TO the amplitude A is of order $10^{-6}-10^{-7}$, so the resulting shifts are small, as assumed. As we will show below, amplitudes in this range can quantitatively fit the dissipation peak but are a factor of 10 too small to fit the period shift of the TO results.

The simple Lorentzian form of these results suggests a strategy for fitting the TO experimental data. Notice that ΔQ^{-1} will have a peak as a function of temperature T if the relaxation time τ passes through the time scale ω_0^{-1} as the temperature is lowered; the peak occurs at T^* such that $\omega_0\tau(T^*)=1$, and at this temperature $\Delta Q^{-1}(T^*)=A$. Therefore, A can be directly determined from the peak value of the dissipation; we can then solve Eq. (12) for $\omega_0\tau$ in terms of $\Delta Q^{-1}(T)/A$ as a function of temperature, allowing us to determine $\tau(T)$. Having determined both A and τ, we can then calculate the period shift $\Delta P/P_0$ as a function of temperature, *with no additional fitting parameters*. In passing we note that the temperature dependence of the shear modulus itself[18] has a much smaller effect on the period shift and dissipation than the temperature dependence of the relaxation time τ.

In Fig. 1 we show the relaxation time obtained from the

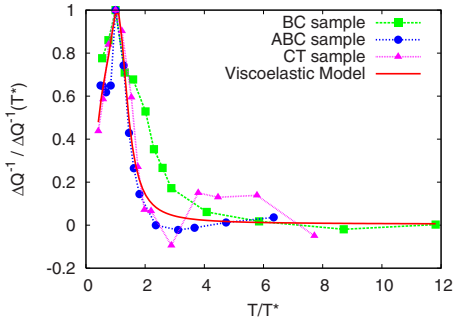

FIG. 2. (Color online) Peaks in the inverse of the quality factor of BeCu TO from Clark *et al.* (Ref. 5). Squares (green) represent the BC sample, circles (blue) represent the ABC sample, and triangles (pink) represent the CT sample. The heavy line (red) is of the viscoelastic model with the change in Q factor (12) and derived relaxation time (14).

measured Q factor of the BeCu TO used in Ref. 5 (to collapse all of the data onto a single curve we have scaled the temperature by the peak temperature T^*). On this log plot we clearly see activated behavior both below and above T^* but with different activation energies. To account for this behavior, we will assume that the relaxation time τ has the functional form

$$\tau(T) = \frac{\tau_0\exp(E_0/T)}{1+\delta\exp(E_1/T)}. \tag{14}$$

For BeCu TO data of Ref. 5 we obtain $\tau_0=260$ ns, $\delta=1.62\times10^{-3}$, $E_0=7.55T^*$, and $E_1=7.01T^*$. At high temperatures, the activation energy for the blocked capillary sample of BeCu TO of Ref. 5 is found to be $E=260.4$ mK (417.5 mK for the annealed blocked capillary sample and 341.4 mK for the constant temperature sample), and at low temperatures $E=18.6$ mK (29.8 and 24.4 mK, respectively). By using the derived relaxation time τ the fits to the dissipation peak and period shift data from Ref. 5 are shown in Figs. 2 and 3. Having fit to the dissipation peak, we see that the same set of parameters accounts for only 10%–20% of the period shift observed in these experiments.

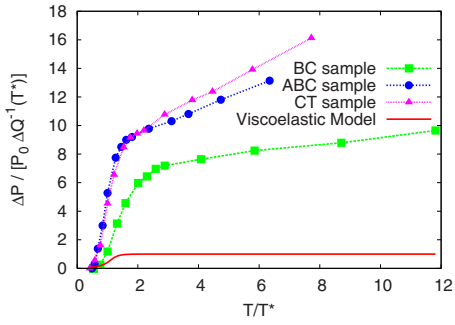

FIG. 3. (Color online) Resonant period shifts of BeCu TO from Clark *et al.* (Ref. 5). Squares (green) represent the BC sample, circles (blue) represent the ABC sample, and triangles (pink) represent the CT sample. The heavy line (red) is of the viscoelastic model with resonant period shift (11) and derived relaxation time (14).

III. CONCLUSION

In summary, we investigated the viscoelastic behavior of solid ^{4}He at low temperatures. The response of a viscoelastic solid to an oscillatory shear stress is determined and used to study the anomalies in the resonant period and the dissipation of the TO experiments. Our approach is quite general, invoking the linear response of the helium together with the simplifying assumption of isotropy; our framework can be used to study other phenomenological models for the mechanical behavior of solid helium. For instance, we could also treat a model with a distribution of relaxation times, such as a "glass" model for the shear modulus[19] of the form $\mu(\omega)=\mu_0(1-i\tau\omega)^{\beta}$. The simple single relaxation time Kelvin-Voigt model identifies a time scale associated with the viscosity of solid ^{4}He; upon lowering the temperature, this relaxation time grows rapidly and eventually passes through ω_0^{-1}, inducing changes in both the dissipation and the oscillator period. While the dissipation peak can be explained completely using the viscoelastic model, the model accounts for only a fraction of the period shift observed in Ref. 5 (although fits to some data[6] yield a smaller discrepancy between the model results and measured period shifts). As originally suggested,[1,2] the remaining period shift may indeed be due to the onset of some type of superfluidity in the solid helium.

ACKNOWLEDGMENTS

The authors would like to thank A. Balatsky, J. Beamish, M. Chan, T. Clark, K. Dasbiswas, P. Goldbart, D. Goswami, M. Graf, D. Huse, H. Kojima, Y. Lee, M. Meisel, N. Mulders, E. Pratt, J. Toner, and J. West for helpful discussions and comments. A.T.D. would like to thank the Aspen Center for Physics, where part of this work was completed. This work was supported by the National Science Foundation.

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